

A RELATIVE VERSION OF THE BEILINSON-HODGE CONJECTURE

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ABSTRACT. Let $k \subseteq \mathbb{C}$ be an algebraically closed subfield, and \mathcal{X} a variety defined over k . One version of the Beilinson-Hodge conjecture that seems to survive scrutiny is the statement that the Betti cycle class map $\mathrm{cl}_{r,m} : H_{\mathcal{M}}^{2r-m}(k(\mathcal{X}), \mathbb{Q}(r)) \rightarrow \mathrm{hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(k(\mathcal{X})(\mathbb{C}), \mathbb{Q}(r)))$ is surjective, that being equivalent to the Hodge conjecture in the case $m = 0$. Now consider a smooth and proper map $\rho : \mathcal{X} \rightarrow \mathcal{S}$ of smooth quasi-projective varieties over k , and where η is the generic point of \mathcal{S} . We anticipate that the corresponding cycle class map is surjective, and provide some evidence in support of this in the case where $\mathcal{X} = \mathcal{S} \times X$ is a product and $m = 1$.

1. INTRODUCTION

The results of this paper are aimed at providing some evidence in support of an affirmative answer to a question first formulated in [SJK-L, Question 1.1], now upgraded to the following:

Conjecture 1.1. *Let $\rho : \mathcal{X} \rightarrow \mathcal{S}$ be a smooth proper map of smooth quasi-projective varieties over a subfield $k = \bar{k} \subseteq \mathbb{C}$, with $\eta = \eta_{\mathcal{S}}$ the generic point of \mathcal{S}/k . Further, let $r, m \geq 0$ be integers. Then*

$\mathrm{cl}_{r,m} : \mathrm{CH}^r(\mathcal{X}_{\eta}, m; \mathbb{Q}) = H_{\mathcal{M}}^{2r-m}(\mathcal{X}_{\eta}, \mathbb{Q}(r)) \rightarrow \mathrm{hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(\mathcal{X}_{\eta}(\mathbb{C}), \mathbb{Q}(r)))$, is surjective.

Here

$$H^{2r-m}(\mathcal{X}_{\eta}(\mathbb{C}), \mathbb{Q}(r)) := \lim_{U \subset \mathcal{S}/k} H^{2r-m}(\rho^{-1}(U)(\mathbb{C}), \mathbb{Q}(r)),$$

is a limit of mixed Hodge structures (MHS), for which one should not expect finite dimensionality, and for any smooth quasi-projective variety W/k , we identify motivic cohomology $H_{\mathcal{M}}^{2r-m}(W, \mathbb{Q}(r))$ with Bloch's higher Chow group $\mathrm{CH}^r(W, m; \mathbb{Q}) := \mathrm{CH}^r(W, m) \otimes \mathbb{Q}$ (see [Bl1]). Note that if $\mathcal{S} = \mathrm{Spec}(k)$, and $m = 0$, then $\mathcal{X} = X_k$ is smooth, projective over k . Thus in this case Conjecture 1.1 reduces to the (classical) Hodge conjecture. The motivation for this conjecture stems from the following:

Firstly, it is a generalization of similar a conjecture in [dJ-L, (§1, statement (S3))], where $\mathcal{X} = \mathcal{S}$, based on a generalization of the Hodge conjecture (classical form) to the higher K -groups, and inspired in part by Beilinson's work in this direction.

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In passing, we hope to instill in the reader that any attempt to deduce Conjecture 1.1 from [dJ-L, §1, statement (S3)] seems to be hopelessly naive, and would require some new technology. To move ahead with this, we eventually work in the special situation where $\mathcal{X} = S \times X$ is a product, with $S = \mathcal{S}$ and X smooth projective, $m = 1$, and employ some motivic input, based on reasonable pre-existing conjectures.

Secondly, as a formal application of M. Saito's theory of mixed Hodge modules (see [A], [K-L], [SJK-L] and the references cited there), one could conceive of the following short exact sequence:

$$\begin{array}{c}
 0 \\
 \downarrow \\
 \text{Ext}_{\text{PMHS}}^1(\mathbb{Q}(0), H^{\nu-1}(\eta_S, R^{2r-\nu-m}\rho_*\mathbb{Q}(r))) \\
 \text{Graded polarizable MHS} \nearrow \\
 \downarrow \\
 (1.2) \quad \left\{ \begin{array}{c} \text{Germs of higher order} \\ \text{generalized normal functions} \end{array} \right\} \\
 \downarrow \\
 \text{hom}_{\text{MHS}}(\mathbb{Q}(0), H^{\nu}(\eta_S, R^{2r-m-\nu}\rho_*\mathbb{Q}(r))) \\
 \downarrow \\
 0
 \end{array}$$

(Warning: As mentioned earlier, passing to the generic point η_S of \mathcal{S} is a limit process, which implies that the spaces above need not be finite dimensional over \mathbb{Q} . This particularly applies to the case $m \geq 1$, where there are residues.) The key point is, is there lurking a generalized Poincaré existence theorem for higher normal functions? Namely, modulo the “fixed part” $\text{Ext}_{\text{PMHS}}^1(\mathbb{Q}(0), H^{\nu-1}(\eta_S, R^{2r-\nu-m}\rho_*\mathbb{Q}(r)))$, are these normal functions cycle-induced? In another direction, this diagram is related to a geometric description of the notion of a Bloch-Beilinson (BB) filtration. As a service to the reader, and to make sense of this all, we elaborate on all of this.

1. For the moment, let us replace $(\eta_S$ by \mathcal{S} , (ν, m) by $(1, 0)$ in diagram (1.2), and where \mathcal{S} is chosen to be a curve). Then this diagram represents the schema of the original Griffiths program aimed at generalizing Lefschetz's famous $(1, 1)$ theorem, via normal functions.¹ This program was aimed at solving the Hodge conjecture inductively. Unfortunately, the lack of a Jacobi inversion theorem for the jacobian

¹Technically speaking, Griffiths worked with normal functions that extended to the boundary $\overline{\mathcal{S}} \setminus \mathcal{S}$, but let's not go there.

of a general smooth projective variety involving a Hodge structure of weight > 1 led to limited applications towards the Hodge conjecture. However the qualitative aspects of his program led to the non-triviality of the now regarded Griffiths group. In that regard, the aforementioned diagram represents a generalization of this idea to the higher K -groups of \mathcal{X} and the general fibers of $\rho : \mathcal{X} \rightarrow \mathcal{S}$.

2. The notion of a BB filtration, first suggested by Bloch and later fortified by Beilinson, tells us that for any X/k smooth projective and $r, m \geq 0$, there should be a descending filtration

$$\{F^\nu \mathrm{CH}^r(X, m; \mathbb{Q}) \mid \nu = 0, \dots, r\},$$

whose graded pieces can be described in terms of extension datum, viz.,

$$Gr_F^\nu \mathrm{CH}^r(X, m; \mathbb{Q}) \simeq \mathrm{Ext}_{\mathcal{MM}}^\nu(\mathrm{Spec}(k), h^{2r-\nu-m}(X)(r)),$$

where \mathcal{MM} is a conjectural category of mixed motives and $h^\bullet(X)(\bullet)$ is motivic cohomology.² Although there were many excellent candidate BB filtrations proposed by others over the years, a few are derived from the point of view of “spreads”, in the case $k = \overline{\mathbb{Q}}$ (see [A], [Lew1], [GG]) as well as a conjectural description in terms of normal functions (see [K-L], [Lew3]). Namely, if X/\mathbb{C} is smooth and projective, then there is a field K of finite transcendence degree over $\overline{\mathbb{Q}}$ and a smooth and proper spread $\mathcal{X} \xrightarrow{\rho} \mathcal{S}$ of smooth quasi-projective varieties over $\overline{\mathbb{Q}}$, such that if η is the generic point of \mathcal{S} , then K can be identified with $\overline{\mathbb{Q}}(\eta)$ via a suitable embedding $\overline{\mathbb{Q}}(\eta) \hookrightarrow \mathbb{C}$; moreover with respect to that embedding, $X/\mathbb{C} = \mathcal{X}_\eta \times_{\overline{\mathbb{Q}}(\eta)} \mathbb{C}$. Diagram (1.2) then provides yet another schema of describing a candidate BB filtration in terms of normal functions.

As indicated earlier, we focus our attention mainly on the case $m = 1$ (K_1 case), and provide some partial results in the case where $\mathcal{X} = S \times X$ is a product, with $S = \mathcal{S}$, and X smooth projective. Our main results are Theorems 4.4, 6.7 and 6.11.

2. NOTATION

- ₀ Unless specified to the contrary, all varieties are defined over \mathbb{C} .
- ₁ $\mathbb{Q}(m)$ is the Tate twist with Hodge type $(-m, -m)$.
- ₂ For a mixed Hodge structure (MHS) H over \mathbb{Q} , we put $\Gamma(H) = \mathrm{hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H)$ and $J(H) = \mathrm{Ext}_{\mathrm{MHS}}^1(\mathbb{Q}(0), H)$.
- ₃ The higher Chow groups $\mathrm{CH}^r(W, m)$ for a quasi-projective variety W over a field k are defined in [Bl1]. Let us assume W/k is regular. An abridged definition of $\mathrm{CH}^r(W, 1)$, viz., in the case $m = 1$ is given by:

$$\mathrm{CH}^r(W, 1) = \frac{\ker \left(\sum_{\mathrm{cd}_W Z=r-1}^Z (k(Z)^\times, Z) \xrightarrow{\mathrm{div}} z^r(W) \right)}{\mathrm{Image of tame symbol}},$$

where $z^r(W)$ is the free abelian group generated by (irreducible) subvarieties of codimension r in W ; moreover, the denominator admits this description. If $V \subset W$

²The original formulation involved only the case $m = 0$; this is just a natural extension of these ideas to the higher K -groups of X .

is an irreducible subvariety of codimension $r - 2$, and $f, g \in k(V)^\times$, then the tame symbol is given as:

$$T(\{f, g\}_V) = \sum_{\text{cd}_V D=1} (-1)^{\nu_D(f)\nu_D(g)} \left(\frac{f^{\nu_D(g)}}{g^{\nu_D(f)}} \right)_D,$$

as D ranges through all irreducible codimension one subvarieties of V , $\nu_D(\cdot)$ is the order of a zero or pole, and $(\cdots)_D$ means the restriction to the generic point of D . The “Image of the tame symbol” is the subgroup generated by $T(\{f, g\}_V)$, as V ranges in W and f, g range through $k(V)^\times$.

•₄ Assume W in •₃ is also smooth of dimension d_W , and let $Z \subset W$ be irreducible of codimension $r - 1$, $f \in k(W)^\times$, where $k \subset \mathbb{C}$ is a subfield. Then the Betti class map

$$\text{cl}_{r,1} : \text{CH}^r(W, 1; \mathbb{Q}) \rightarrow \Gamma(H^{2r-1}(W, \mathbb{Q}(r))) \subset H^{2r-1}(W, \mathbb{Q}(r)) \simeq [H_c^{2d_W-2r+1}(W, \mathbb{Q}(d_W-r))]^\vee \subset [H_c^{2d_W-2r+1}(W, \mathbb{C})]^\vee$$

is induced by the current

$$(Z, f) \mapsto \frac{1}{(2\pi i)^{d_W-r+1}} \int_Z d \log(f) \wedge \omega, \quad \{\omega\} \in H_c^{2d_W-2r+1}(W, \mathbb{C}).$$

3. WHAT IS KNOWN

In this section, we summarize some of the results in [SJK-L], where $r \geq m = 1$. The setting is the following diagram

$$\begin{array}{ccc} \mathcal{X}^\mathbb{C} & \xrightarrow{\quad} & \overline{\mathcal{X}} \\ \downarrow \rho & & \downarrow \overline{\rho} \\ \mathcal{S}^\mathbb{C} & \xrightarrow{\quad} & \overline{\mathcal{S}} \end{array}$$

where $\overline{\mathcal{X}}$ and $\overline{\mathcal{S}}$ are nonsingular complex projective varieties, $\overline{\rho}$ is a dominating flat morphism, $D \subset \overline{\mathcal{S}}$ a divisor, $\mathcal{Y} := \overline{\rho}^{-1}(D)$, $\mathcal{S} := \overline{\mathcal{S}} \setminus D$, $\mathcal{X} := \overline{\mathcal{X}} \setminus \mathcal{Y}$ and $\rho := \overline{\rho}|_{\mathcal{X}}$.³ There is a short exact sequence

$$(3.1) \quad 0 \rightarrow \frac{H^{2r-1}(\overline{\mathcal{X}}, \mathbb{Q}(r))}{H_y^{2r-1}(\overline{\mathcal{X}}, \mathbb{Q}(r))} \rightarrow H^{2r-1}(\mathcal{X}, \mathbb{Q}(r)) \rightarrow H_y^{2r}(\overline{\mathcal{X}}, \mathbb{Q}(r))^\circ \rightarrow 0,$$

where with regard to the former term in (3.1), $H_y^{2r-1}(\overline{\mathcal{X}}, \mathbb{Q}(r))$ is identified with its image in $H^{2r-1}(\overline{\mathcal{X}}, \mathbb{Q}(r))$, and $H_y^{2r}(\overline{\mathcal{X}}, \mathbb{Q}(r))^\circ := \ker(H_y^{2r}(\overline{\mathcal{X}}, \mathbb{Q}(r)) \rightarrow H^{2r}(\overline{\mathcal{X}}, \mathbb{Q}(r)))$. One has a corresponding diagram

$$(3.2) \quad \begin{array}{ccccc} \text{CH}^r(\mathcal{X}, 1; \mathbb{Q}) & \xrightarrow{\quad} & \text{CH}_y^r(\overline{\mathcal{X}}; \mathbb{Q})^\circ & \xrightarrow{\alpha_y} & \text{CH}_{\text{hom}}^r(\overline{\mathcal{X}}; \mathbb{Q}) \\ \downarrow \text{cl}_{r,1}^\mathcal{X} & & \downarrow \beta_y & & \downarrow \underline{AJ}^{\overline{\mathcal{X}}} \\ \Gamma(H^{2r-1}(\mathcal{X}, \mathbb{Q}(r))) & \hookrightarrow & \Gamma(H_y^{2r}(\overline{\mathcal{X}}, \mathbb{Q}(r))^\circ) & \longrightarrow & J\left(\frac{H^{2r-1}(\overline{\mathcal{X}}, \mathbb{Q}(r))}{H_y^{2r-1}(\overline{\mathcal{X}}, \mathbb{Q}(r))}\right) \end{array}$$

well-known to commute by an extension class argument, and where $\underline{AJ}^{\overline{\mathcal{X}}}$ is the corresponding “reduced” Abel-Jacobi map. Further, the definition of $\text{CH}_y^r(\overline{\mathcal{X}}; \mathbb{Q})^\circ$

³While we assume that the base field is \mathbb{C} , the results here are valid for varieties over an algebraically closed field $k \subset \mathbb{C}$.

is the obvious one, being the cycles in $\mathrm{CH}^{r-1}(\mathcal{Y}; \mathbb{Q})$ that are homologous to zero on $\overline{\mathcal{X}}$.

Remark 3.3. *Poincaré duality gives an isomorphism of MHS:*

$$H_{\mathcal{Y}}^{2r}(\overline{\mathcal{X}}, \mathbb{Q}(r)) \simeq H_{2 \dim \overline{\mathcal{X}} - 2r}(\mathcal{Y}, \mathbb{Q}(\dim \overline{\mathcal{X}} - r)).$$

Thus $\ker \beta_{\mathcal{Y}} = \mathrm{CH}_{\mathrm{hom}}^{r-1}(\mathcal{Y}; \mathbb{Q})$, viz., the subspace of cycles in $\mathrm{CH}^{r-1}(\mathcal{Y}; \mathbb{Q})$ that are homologous to zero on \mathcal{Y} .

Let us assume that $\beta_{\mathcal{Y}}$ is surjective, as is the case if the (classical) Hodge conjecture holds. If we apply the snake lemma, we arrive at

$$\mathrm{coker}(\mathrm{cl}_{r,1}^{\overline{\mathcal{X}}}) \simeq \frac{\ker \left[\underline{AJ}^{\overline{\mathcal{X}}} \big|_{\mathrm{Image}(\alpha_{\mathcal{Y}})} : \mathrm{Image}(\alpha_{\mathcal{Y}}) \rightarrow J \left(\frac{H_{\mathcal{Y}}^{2r-1}(\overline{\mathcal{X}}, \mathbb{Q}(r))}{H_{\mathcal{Y}}^{2r-1}(\overline{\mathcal{X}}, \mathbb{Q}(r))} \right) \right]}{\alpha_{\mathcal{Y}}(\ker(\beta_{\mathcal{Y}}))}.$$

Now take the limit over all $D \subset \overline{\mathcal{S}}$ to arrive at an induced cycle map:

$$(3.4) \quad \mathrm{cl}_{r,1}^{\eta} : \mathrm{CH}^r(\mathcal{X}_{\eta}, 1; \mathbb{Q}) \rightarrow \Gamma(H^{2r-1}(\mathcal{X}_{\eta}, \mathbb{Q}(r))).$$

where η is the generic point of $\overline{\mathcal{S}}$. We arrive at:

$$(3.5) \quad \frac{\Gamma(H^{2r-1}(\mathcal{X}_{\eta}, \mathbb{Q}(r)))}{\mathrm{cl}_{r,1}^{\eta}(\mathrm{CH}^r(\mathcal{X}_{\eta}, 1; \mathbb{Q}))} \simeq \frac{\ker \left[\mathcal{K} \xrightarrow{AJ} J \left(\frac{H_{\mathcal{S}}^{2r-1}(\overline{\mathcal{X}}, \mathbb{Q}(r))}{N_{\mathcal{S}}^1 H^{2r-1}(\overline{\mathcal{X}}, \mathbb{Q}(r))} \right) \right]}{N_{\mathcal{S}}^1 \mathrm{CH}^r(\overline{\mathcal{X}}; \mathbb{Q})},$$

where $\mathcal{K} := \ker[\mathrm{CH}_{\mathrm{hom}}^r(\overline{\mathcal{X}}; \mathbb{Q}) \rightarrow \mathrm{CH}^r(\mathcal{X}_{\eta}; \mathbb{Q})]$, $N_{\mathcal{S}}^q \mathrm{CH}^r(\overline{\mathcal{X}}) \subset \mathrm{CH}_{\mathrm{hom}}^r(\overline{\mathcal{X}}; \mathbb{Q})$ is the subgroup generated by cycles which are homologous to zero on some codimension q subscheme of $\overline{\mathcal{X}}$ obtained from a (pure) codimension q subscheme of $\overline{\mathcal{S}}$ via $\overline{\rho}^{-1}$ (keep in mind Remark 3.3, in the case $q = 1$), and $N_{\mathcal{S}}^q H^{2r-1}(\overline{\mathcal{X}}, \mathbb{Q}(r))$ is the subspace of the coniveau $N^q H^{2r-1}(\overline{\mathcal{X}}, \mathbb{Q}(r))$ arising from q codimensional subschemes of $\overline{\mathcal{S}}$ via $\overline{\rho}^{-1}$. A relatively simple argument, found in [SJK-L], yields the following:

Proposition 3.6. *Under the assumption of the Hodge conjecture, (3.5) becomes:*

$$\frac{\Gamma(H^{2r-1}(\mathcal{X}_{\eta}, \mathbb{Q}(r)))}{\mathrm{cl}_{r,1}^{\eta}(\mathrm{CH}^r(\mathcal{X}_{\eta}, 1; \mathbb{Q}))} \simeq \frac{N_{\mathcal{S}}^1 \mathrm{CH}^r(\overline{\mathcal{X}}; \mathbb{Q}) + \ker [\mathcal{K} \xrightarrow{AJ} J(H^{2r-1}(\overline{\mathcal{X}}, \mathbb{Q}(r)))]}{N_{\mathcal{S}}^1 \mathrm{CH}^r(\overline{\mathcal{X}}; \mathbb{Q})}.$$

Example 3.7. *Suppose that $\overline{\mathcal{X}} = \overline{\mathcal{S}}$ with $\overline{\rho}$ the identity. In this case Proposition 3.6 becomes:*

$$(3.8) \quad \frac{\Gamma(H^{2r-1}(\mathbb{C}(\overline{\mathcal{X}}), \mathbb{Q}(r)))}{\mathrm{cl}_{r,1}(\mathrm{CH}^r(\mathrm{Spec}(\mathbb{C}(\overline{\mathcal{X}})), 1; \mathbb{Q}))} \simeq \frac{N^1 \mathrm{CH}^r(\overline{\mathcal{X}}; \mathbb{Q}) + \mathrm{CH}_{AJ}^r(\overline{\mathcal{X}}; \mathbb{Q})}{N^1 \mathrm{CH}^r(\overline{\mathcal{X}}; \mathbb{Q})},$$

where $N^1 \mathrm{CH}^r(\overline{\mathcal{X}}; \mathbb{Q})$ is the subgroup of cycles, that are homologous to zero on codimension 1 subschemes of $\overline{\mathcal{X}}$, and $\mathrm{CH}_{AJ}^r(\overline{\mathcal{X}}; \mathbb{Q})$ are cycles in the kernel of the Abel-Jacobi map $AJ : \mathrm{CH}^r(\overline{\mathcal{X}}; \mathbb{Q}) \rightarrow J(H^{2r-1}(\overline{\mathcal{X}}, \mathbb{Q}(r)))$. According to Jannsen [Ja1, p. 227], there is a discussion that strongly hints that the right hand side of (3.8) should be zero. In light of [Lew2], we conjecturally believe this to be true. In particular, since $\mathrm{Spec}(\mathbb{C}(\overline{\mathcal{X}}))$ is a point, this implies that $\Gamma(H^{2r-1}(\mathbb{C}(\overline{\mathcal{X}}), \mathbb{Q}(r))) = 0$ for $r > 1$. The reader can easily check that

$$\mathrm{cl}_{r,1}(\mathrm{CH}^r(\mathrm{Spec}(\mathbb{C}(\overline{\mathcal{X}})), 1; \mathbb{Q})) = \Gamma(H^{2r-1}(\mathbb{C}(\overline{\mathcal{X}}), \mathbb{Q}(r))),$$

holds unconditionally in the case $r = \dim \overline{\mathcal{X}}$, that being well known in the case $r = \dim \mathcal{X} = 1$, and for $r = \dim \mathcal{X} > 1$, from the weak Lefschetz theorem for affine varieties.

Example 3.9. [SJK-L] Here we give some evidence that the RHS (hence LHS) of Proposition 3.6 is zero. Suppose $\overline{\mathcal{X}} = X \times \overline{S}$, ($\overline{S} := \overline{S}$), and let us assume the condition

$$\mathrm{CH}^r(\overline{\mathcal{X}}; \mathbb{Q}) = \bigoplus_{\ell=0}^r \mathrm{CH}^\ell(\overline{S}; \mathbb{Q}) \otimes \mathrm{CH}^{r-\ell}(X; \mathbb{Q}).$$

An example situation is when \overline{S} is a flag variety, such as a projective space; however conjecturally speaking, this condition is expected to hold for a much broader class of examples. If we further assume the Hodge conjecture, then as a consequence of Proposition 3.6, we arrive at:

Corollary 3.10. Under the assumptions on the Künneth condition above and the Hodge conjecture, with $\overline{\mathcal{X}} = \overline{S} \times X$, if

$$\mathrm{cl}_{r,1}(\mathrm{CH}^r(\mathrm{Spec}(\mathbb{C}(\overline{S})), 1; \mathbb{Q})) = \Gamma(H^{2r-1}(\mathbb{C}(\overline{S}), \mathbb{Q}(r))),$$

holds, then the map $\mathrm{cl}_{r,1}^\eta$ in (3.4) is surjective.

4. THE SPLIT CASE AND RIGIDITY

4.1. Base a curve. In this section, we observe that the Beilinson-Hodge conjecture (Conjecture 1.1), in the special case of a split projection with base given by a curve, holds under the assumption of the Hodge conjecture on the fibre. Let X be a smooth projective variety and C a smooth curve. Let $\pi : C \times X \rightarrow C$ denote the projection morphism.

Proposition 4.1. Let X and C be as above.

- (1) If $m > 1$, then $\mathrm{CH}^r(C \times X, m) \rightarrow \Gamma(H^{2r-m}(C \times X, \mathbb{Q}(r)))$ is surjective.
- (2) If $m = 1$, then $\mathrm{CH}^r(C \times X, m) \rightarrow \Gamma(H^{2r-m}(C \times X, \mathbb{Q}(r)))$ is surjective if the Hodge conjecture holds for X in codimension $r - 1$.

In particular, the Beilinson Hodge conjecture (Conjecture 1.1) for $\pi : C \times X \rightarrow C$ holds unconditionally if $m > 1$ and, if $m = 1$, then it holds under the assumption of the Hodge conjecture for X .

We begin with some preliminary reductions. By the Künneth decomposition we can identify $H^{2r-m}(C \times X, \mathbb{Q})$ with

$$\bigoplus_{i=0}^2 H^i(C, \mathbb{Q}) \otimes H^{2r-m-i}(X, \mathbb{Q}).$$

For $i = 0$ and 2 , $H^i(C, \mathbb{Q}) \otimes H^{2r-m-i}(X, \mathbb{Q})(r)$ is pure of weight $-m$ by the purity of $H^{2r-m-i}(X, \mathbb{Q})$ and that of $H^0(C, \mathbb{Q})$ as well as $H^2(C, \mathbb{Q})$. Therefore $\Gamma(H^i(C, \mathbb{Q}) \otimes H^{2r-m-i}(X, \mathbb{Q})(r)) = 0$ for $m > 0$ and $i \neq 1$. The same also holds for $i = 1$ if C is projective. But in general we have the following.

Lemma 4.2. With notation and assumptions as above,

$$\Gamma(H^1(C, \mathbb{Q}) \otimes H^{2r-2}(X, \mathbb{Q})(r)) = \Gamma(H^1(C, \mathbb{Q}(1))) \otimes \Gamma(H^{2r-2}(X, \mathbb{Q}(r-1))).$$

Proof. Let H denote $H^{2r-2}(X, \mathbb{Q}(r-1))$. Setting $W_j = W_j H^1(C, \mathbb{Q}(1))$ gives the following commutative diagram of mixed Hodge structures with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & W_{-1} \otimes H & \longrightarrow & W_0 \otimes H & \longrightarrow & Gr_0^W \otimes H \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & W_{-1} \otimes \Gamma(H) & \longrightarrow & W_0 \otimes \Gamma(H) & \longrightarrow & Gr_0^W \otimes \Gamma(H) \longrightarrow 0 \end{array}$$

Since $W_{-1} \otimes H$ and $W_{-1} \otimes \Gamma(H)$ have negative weights, their Γ 's are trivial. It follows that we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(W_0 \otimes H) & \longrightarrow & \Gamma(Gr_0^W \otimes H) & \longrightarrow & J(W_{-1} \otimes H) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \Gamma(W_0 \otimes \Gamma(H)) & \longrightarrow & \Gamma(Gr_0^W \otimes \Gamma(H)) & \longrightarrow & J(W_{-1} \otimes \Gamma(H)) \end{array}$$

Since Gr_0^W is pure Tate of weight 0, the middle vertical is an isomorphism. We also have an injection $H^1(\overline{C}, \mathbb{Q}) \rightarrow H^1(C, \mathbb{Q})$, identifying W_{-1} with $H^1(\overline{C}, \mathbb{Q}(1))$. Using semi-simplicity of polarized Hodge structures we see that the natural injection

$$W_{-1} \otimes \Gamma(H) \rightarrow W_{-1} \otimes H$$

is split, so that we obtain a split injection

$$J(W_{-1} \otimes \Gamma(H)) \rightarrow J(W_{-1} \otimes H) .$$

Finally, noting that $\Gamma(W_0 \otimes \Gamma(H)) = \Gamma(W_0) \otimes \Gamma(H)$ and using the snake lemma gives the desired result. \square

Proof of Proposition 4.1. If $m > 1$, then by a weight argument

$$\Gamma(H^1(C, \mathbb{Q}) \otimes H^{2r-m-1}(X, \mathbb{Q})(r)) = 0$$

and the surjectivity is trivial. For r arbitrary, and $m = 1$, first note that $\text{CH}^1(C, 1)$ maps surjectively to $\Gamma(H^1(C, \mathbb{Q}(1)))$. Under the assumption of the Hodge conjecture for X in codimension $r - 1$, one also has that

$$\text{CH}^{r-1}(X, 0; \mathbb{Q}) \rightarrow \Gamma(H^{2r-2}(X, \mathbb{Q}(r-1)))$$

is surjective. Since the natural morphism

$$\text{CH}^1(C, 1) \otimes \text{CH}^{r-1}(X, 0) \rightarrow \text{CH}^r(C \times X, 1)$$

induced by pullback to $C \times X$ followed by the cup product is compatible with the tensor product in the Künneth decomposition, the previous remarks and Lemma 4.2 show that $\text{cl}_{1,r}$ surjects onto $\Gamma(H^1(C, \mathbb{Q}) \otimes H^{2r-2}(X, \mathbb{Q})(r-1))$. The last claim in the proposition follows from the first two by taking limits over open $U \subset C$. \square

Remark 4.3. Note that in the situation above, the surjectivity in Conjecture 1.1 holds for every open $U \subset C$, and, in particular, one does not need to pass to the generic point. However, in general, in the examples of [dJ-L, Section 5] one has to.

4.2. Base a product of two curves. In this section, we prove the strong form (i.e., without passing to the generic point) of the Beilinson-Hodge conjecture (Conjecture 1.1) for $r = 2$ and $m = 1$ in the special case of a split projection with base given by a product of two curves, under a certain rigidity assumption (see below). More precisely, let X be smooth projective, and $\overline{C}_1, \overline{C}_2$ smooth projective curves, with non-empty open $C_j \subsetneq \overline{C}_j$. Let $\overline{S} = \overline{C}_1 \times \overline{C}_2$, $S = C_1 \times C_2$, $\Sigma_j = \overline{C}_j \setminus C_j$, and $E = \overline{S} \setminus S = \Sigma_1 \times \overline{C}_2 \cup \overline{C}_1 \times \Sigma_2$. Finally, let $\mathcal{X} = S \times X$ and let $\pi : \mathcal{X} \rightarrow S$ denote the canonical projection map.

Theorem 4.4. *Let X and S be as above. If $H^2(\overline{S}, \mathbb{Q})$ does not have a non-zero \mathbb{Q} subHodge structure contained in $H^{2,0}(\overline{S}) \oplus H^{0,2}(\overline{S})$, then*

$$\text{cl}_{2,1} : \text{CH}^2(S \times X, 1; \mathbb{Q}) \rightarrow \Gamma(H^3(S \times X, \mathbb{Q}(2)))$$

is surjective.

Note that if we were to replace S with $C_1 \times \overline{C}_2$ or $S = C_1 \times \overline{C}_2$ then the result is already part of Proposition 4.1. Again this holds even if the C_i are complete by reduction to the case of one curve in the base.

We begin with some preliminary reductions. First observe that, as $\dim S = 2$, by the weak Lefschetz theorem for affine varieties, the Künneth decomposition of $H^3(S \times X, \mathbb{Q})$ is

$$H^0(S, \mathbb{Q}) \otimes H^3(X, \mathbb{Q}) \oplus H^1(S, \mathbb{Q}) \otimes H^2(X, \mathbb{Q}) \oplus H^2(S, \mathbb{Q}) \otimes H^1(X, \mathbb{Q}),$$

and we shall deal with the three summands separately (after twisting with $\mathbb{Q}(2)$).

For the first term, note that $H^0(S, \mathbb{Q}) \otimes H^3(X, \mathbb{Q})(2)$ is pure of weight -1 by the purity of $H^3(X, \mathbb{Q})$, so that $\Gamma(H^0(S, \mathbb{Q}) \otimes H^3(X, \mathbb{Q})(2)) = 0$.

Lemma 4.5. *With notation as above,*

$$\Gamma(H^1(S, \mathbb{Q}) \otimes H^2(X, \mathbb{Q})(2)) = \Gamma(H^1(S, \mathbb{Q}(1))) \otimes \Gamma(H^2(X, \mathbb{Q}(1)))$$

and $\text{cl}_{1,2}$ is surjective.

Proof. The proof of the equality is the same as the proof of Lemma 4.2. Furthermore, the proof of surjectivity of $\text{cl}_{1,2}$ is similar to the proof of Proposition 4.1. We leave the details to the reader. \square

For the proof of Theorem 4.4 it remains to show that the image of $\text{cl}_{1,2}$ contains $\Gamma(H^2(S, \mathbb{Q}) \otimes H^1(X, \mathbb{Q})(2))$. The following lemma gives a description of the latter in terms of the Abel-Jacobi map. First put

$$H_E^3(\overline{S}, \mathbb{Q})^\circ = \ker [H_E^3(\overline{S}, \mathbb{Q}) \rightarrow H^3(\overline{S}, \mathbb{Q})].$$

Lemma 4.6. *One has these identifications:*

$$(i) \Gamma(H^2(S, \mathbb{Q}) \otimes H^1(X, \mathbb{Q})(2)) \simeq$$

$$\ker [\Gamma(H_E^3(\overline{S}, \mathbb{Q}(2))^\circ \otimes H^1(X, \mathbb{Q})) \rightarrow J(H^1(\overline{C}_1, \mathbb{Q}(1)) \otimes H^1(\overline{C}_2, \mathbb{Q}(1)) \otimes H^1(X, \mathbb{Q}))].$$

$$(ii) \Gamma(H_E^3(\overline{S}, \mathbb{Q}(2))^\circ \otimes H^1(X, \mathbb{Q})) \simeq$$

$$\Gamma\left([H^1(\overline{C}_1, \mathbb{Q}) \otimes H_{\deg 0}^0(\Sigma_2, \mathbb{Q}) \bigoplus H_{\deg 0}^0(\Sigma_1, \mathbb{Q}) \otimes H^1(\overline{C}_2, \mathbb{Q})] \otimes H^1(X, \mathbb{Q}(1))\right).$$

Proof. Part (i): Observe that

$$\frac{H^2(\overline{S}, \mathbb{Q})}{H_E^2(\overline{S}, \mathbb{Q})} = H^1(\overline{C}_1, \mathbb{Q}) \otimes H^1(\overline{C}_2, \mathbb{Q}), \quad H^2(S, \mathbb{Q}) = H^1(C_1, \mathbb{Q}) \otimes H^1(C_2, \mathbb{Q}),$$

as $H^2(C_j) = 0$. There is a short exact sequence:

$$0 \rightarrow \frac{H^2(\overline{S}, \mathbb{Q})}{H_E^2(\overline{S}, \mathbb{Q})} \rightarrow H^2(S, \mathbb{Q}) \rightarrow H_E^3(\overline{S}, \mathbb{Q})^\circ \rightarrow 0.$$

This in turn gives rise to a short exact sequence:

$$0 \rightarrow H^1(\overline{C}_1, \mathbb{Q}(1)) \otimes H^1(\overline{C}_2, \mathbb{Q}(1)) \otimes H^1(X, \mathbb{Q}) \rightarrow H^1(C_1, \mathbb{Q}(1)) \otimes H^1(C_2, \mathbb{Q}(1)) \otimes H^1(X, \mathbb{Q}) \rightarrow H_E^3(\overline{S}, \mathbb{Q}(2))^\circ \otimes H^1(X, \mathbb{Q}) \rightarrow 0.$$

So, using purity,

$$\Gamma(H^1(C_1, \mathbb{Q}(1)) \otimes H^1(C_2, \mathbb{Q}(1)) \otimes H^1(X, \mathbb{Q}))$$

can be identified with

$$\ker [\Gamma(H_E^3(\overline{S}, \mathbb{Q}(2))^\circ \otimes H^1(X, \mathbb{Q})) \rightarrow J(H^1(\overline{C}_1, \mathbb{Q}(1)) \otimes H^1(\overline{C}_2, \mathbb{Q}(1)) \otimes H^1(X, \mathbb{Q}))].$$

Part (ii): Poincaré duality gives an isomorphism of MHS

$$(4.7) \quad H_E^3(\overline{S}, \mathbb{Q}(2)) \simeq H_1(E, \mathbb{Q}), \quad \text{hence } H_E^3(\overline{S}, \mathbb{Q})^\circ \simeq \ker (H_1(E, \mathbb{Q}) \rightarrow H_1(\overline{S}, \mathbb{Q})).$$

Moreover the Mayer-Vietoris sequence gives us the exact sequence

$$0 \rightarrow [H_1(\overline{C}_1 \times \Sigma_2, \mathbb{Q}) \oplus H_1(\Sigma_1 \times \overline{C}_2, \mathbb{Q})] \otimes H^1(X, \mathbb{Q}) \rightarrow H_1(E, \mathbb{Q}) \otimes H^1(X, \mathbb{Q}) \rightarrow H_0(\Sigma_1 \times \Sigma_2, \mathbb{Q}) \otimes H^1(X, \mathbb{Q}).$$

But $\Gamma(H_0(\Sigma_1 \times \Sigma_2, \mathbb{Q}) \otimes H^1(X, \mathbb{Q})) = 0$; moreover one has a commutative diagram

$$\begin{array}{ccc} H_1(\overline{C}_1 \times \Sigma_2, \mathbb{Q}) \oplus H_1(\Sigma_1 \times \overline{C}_2, \mathbb{Q}) & \hookrightarrow & H_1(E, \mathbb{Q}) \\ \parallel & & \downarrow \\ H_1(\overline{C}_1, \mathbb{Q}) \otimes H_0(\Sigma_2, \mathbb{Q}) \oplus H_0(\Sigma_1, \mathbb{Q}) \otimes H_1(\overline{C}_2, \mathbb{Q}) & & \\ \downarrow & & \\ H_1(\overline{C}_1, \mathbb{Q}) \otimes H_0(\overline{C}_2, \mathbb{Q}) \oplus H_0(\overline{C}_1, \mathbb{Q}) \otimes H_1(\overline{C}_2, \mathbb{Q}) & \simeq & H_1(\overline{S}, \mathbb{Q}). \end{array}$$

Hence from this and (4.7), $\Gamma(H_E^3(\overline{S}, \mathbb{Q}(2))^\circ \otimes H^1(X, \mathbb{Q}))$ can be identified with

$$\Gamma\left([H^1(\overline{C}_1, \mathbb{Q}) \otimes H_{\deg 0}^0(\Sigma_2, \mathbb{Q}) \bigoplus H_{\deg 0}^0(\Sigma_1, \mathbb{Q}) \otimes H^1(\overline{C}_2, \mathbb{Q})] \otimes H^1(X, \mathbb{Q}(1))\right).$$

□

Proof of Theorem 4.4. Note that by the Lefschetz (1,1) theorem, $\Gamma(H^1(\overline{C}_j, \mathbb{Q}) \otimes H^1(X, \mathbb{Q})(1))$ is algebraic. Let us assume for the moment that there exists B in $\Gamma(H_{\deg 0}^0(\Sigma_1, \mathbb{Q}) \otimes H^1(\overline{C}_2, \mathbb{Q}) \otimes H^1(X, \mathbb{Q}(1)))$ of the form $B = \xi \times D$, where $D \subset \overline{C}_2 \times X$ is an irreducible curve, $\xi \in H_{\deg 0}^0(\Sigma_1, \mathbb{Q})$, and that B is in the kernel of the Abel-Jacobi map in Lemma 4.6(i). Notice that the inclusion $H^1(\overline{C}_1, \mathbb{Q}(1)) \otimes \mathbb{Q}(1) \cdot [D] \hookrightarrow H^1(\overline{C}_1, \mathbb{Q}(1)) \otimes H^1(\overline{C}_2, \mathbb{Q}(1)) \otimes H^1(X, \mathbb{Q})$ defines a splitting, and hence an inclusion

$$J(H^1(\overline{C}_1, \mathbb{Q}(1))) \hookrightarrow J(H^1(\overline{C}_1, \mathbb{Q}(1)) \otimes H^1(\overline{C}_2, \mathbb{Q}(1)) \otimes H^1(X, \mathbb{Q})).$$

By applying Abel's theorem to \overline{C}_1 , it follows that there exists $f \in \mathbb{C}(\overline{C}_1 \times D)^\times$ for which $(f) = \xi \times D = B$, thus supplying the necessary element in $\text{CH}^2(S \times X, 1; \mathbb{Q})$.

The same story holds if we replace D by any divisor with non-trivial image in the Neron-Severi group. Using a basis for the Neron-Severi group of $\overline{\mathcal{C}}_2 \times X$, one sees that the kernel of the Abel-Jacobi map restricted to $\Gamma(H_{\deg 0}^0(\Sigma_1, \mathbb{Q}) \otimes H^1(\overline{\mathcal{C}}_2, \mathbb{Q}) \otimes H^1(X, \mathbb{Q}(1)))$ is in the image of $\text{cl}_{1,2}$. A similar story holds separately for A in $\Gamma(H^1(\overline{\mathcal{C}}_1, \mathbb{Q}) \otimes H_{\deg 0}^0(\Sigma_2, \mathbb{Q}) \otimes H^1(X, \mathbb{Q}(1)))$ in the kernel of the Abel-Jacobi map.

The more complicated issue is the case where $A + B$ in

$$\Gamma\left([H^1(\overline{\mathcal{C}}_1, \mathbb{Q}) \otimes H_{\deg 0}^0(\Sigma_2, \mathbb{Q}) \bigoplus H_{\deg 0}^0(\Sigma_1, \mathbb{Q}) \otimes H^1(\overline{\mathcal{C}}_2, \mathbb{Q})] \otimes H^1(X, \mathbb{Q}(1))\right)$$

is in the kernel of the Abel-Jacobi map. The problem boils down to the following. There are two subHodge structures V_1, V_2 of $H^1(\overline{\mathcal{C}}_1, \mathbb{Q}(1)) \otimes H^1(\overline{\mathcal{C}}_2, \mathbb{Q}(1)) \otimes H^1(X, \mathbb{Q})$, where $V_1 = H^1(\overline{\mathcal{C}}_1, \mathbb{Q}(1)) \otimes \Gamma(H^1(\overline{\mathcal{C}}_2, \mathbb{Q}(1)) \otimes H^1(X, \mathbb{Q}))$, and $V_2 \simeq H^1(\overline{\mathcal{C}}_2, \mathbb{Q}(1)) \otimes \Gamma(H^1(\overline{\mathcal{C}}_1, \mathbb{Q}(1)) \otimes H^1(X, \mathbb{Q}))$ is defined similarly. If their intersection V is trivial, then

$$J(V_1) \oplus J(V_2) \hookrightarrow J(H^1(\overline{\mathcal{C}}_1, \mathbb{Q}(1)) \otimes H^1(\overline{\mathcal{C}}_2, \mathbb{Q}(1)) \otimes H^1(X, \mathbb{Q})),$$

so $A + B$ in the kernel of the Abel-Jacobi map implies that A, B are in the kernel, and from our earlier discussion it follows that then

$$\text{cl}_{2,1} : \text{CH}^2(S \times X, 1; \mathbb{Q}) \rightarrow \Gamma H^3(S \times X, \mathbb{Q}(2))$$

is surjective. If V is non-trivial, then from types we see that $V(-2)$ is contained in

$$\{H^{1,0}(\overline{\mathcal{C}}_1) \otimes H^{1,0}(\overline{\mathcal{C}}_2) \otimes H^{0,1}(X)\} \bigoplus \{H^{0,1}(\overline{\mathcal{C}}_1) \otimes H^{0,1}(\overline{\mathcal{C}}_2) \otimes H^{1,0}(X)\}$$

inside $H^2(\overline{\mathcal{S}}, \mathbb{Q}) \otimes H^1(X, \mathbb{Q})$. Tensoring this with $H^{2d-1}(X, \mathbb{Q}(d))$, where $d = \dim X$, and applying the cup product,

$$H^1(X, \mathbb{Q}) \times H^{2d-1}(X, \mathbb{Q}(d)) \xrightarrow{\cup} H^{2d}(X, \mathbb{Q}(d)),$$

followed by the identification $H^{2d}(X, \mathbb{Q}(d)) \simeq \mathbb{Q}$, we find that $V(-2)$ results in a non-trivial \mathbb{Q} -subHodge structure of $H^2(\overline{\mathcal{S}}, \mathbb{Q})$ contained in $H^{2,0}(\overline{\mathcal{S}}) \oplus H^{0,2}(\overline{\mathcal{S}})$. \square

We conclude this section with some discussion of the rigidity condition appearing in Theorem 4.4.

Example 4.8. *If the $\overline{\mathcal{C}}_j$ are elliptic curves, then the existence of a non-trivial (hence rank 2) \mathbb{Q} -subHodge structure implies that $\overline{\mathcal{C}}_j = \mathbb{C}/\{\mathbb{Z} \oplus \mathbb{Z}\tau_j\}$, where $\mathbb{Q}(\tau_j)/\mathbb{Q}$ is a quadratic extension. To see this, observe that this subHodge structure by Poincaré duality corresponds to two independent classes in $H_2(\overline{\mathcal{C}}_1 \times \overline{\mathcal{C}}_2, \mathbb{Q})$,*

$$\xi_j = k_1^{(j)} \alpha_1 \otimes \alpha_2 + k_2^{(j)} \alpha_1 \otimes \beta_2 + k_3^{(j)} \beta_1 \otimes \alpha_2 + k_4^{(j)} \beta_1 \otimes \beta_2 \quad (j = 1, 2),$$

where $\alpha_j = \overline{[0, 1]}$, $\beta_j = \overline{[0, \tau_j]}$, and all $k_i^{(j)}$ are in \mathbb{Q} . Thus because of types we have

$$0 = \int_{\xi_j} dz_1 \wedge d\overline{z}_2 = k_1^{(j)} + k_2^{(j)} \overline{\tau}_2 + k_3^{(j)} \tau_1 + k_4^{(j)} \tau_1 \overline{\tau}_2 \quad (j = 1, 2),$$

so that

$$\tau_1 = -\frac{k_2^{(j)} \overline{\tau}_2 + k_1^{(j)}}{k_4^{(j)} \overline{\tau}_2 + k_3^{(j)}} \quad (j = 1, 2).$$

Using $j = 1, 2$, we can solve for τ_2 , viz.,

$$[k_2^{(1)} k_4^{(2)} - k_2^{(2)} k_4^{(1)}] \tau_2^2 + \dots = 0.$$

But ξ_1, ξ_2 independent and $\text{Im}(\tau_j) \neq 0$ implies that $k_2^{(1)}k_4^{(2)} - k_2^{(2)}k_4^{(1)} \neq 0$. Then $\mathbb{Q}(\tau_1) = \mathbb{Q}(\overline{\tau}_2) = \mathbb{Q}(\tau_2)$, so that the \overline{C}_j are isogenous and have complex multiplication. Therefore the Neron-Severi group of $\overline{C}_1 \times \overline{C}_2$ has rank 4, and gives rise to a subHodge structure $W \subseteq H^2(\overline{C}_1 \times \overline{C}_2, \mathbb{Q}(1))$ of dimension 4, with $W_{\mathbb{C}} = H^{1,1}(\overline{C}_1 \times \overline{C}_2)$. Its orthogonal V under the cup product $H^2(\overline{C}_1 \times \overline{C}_2, \mathbb{Q}(1)) \otimes H^2(\overline{C}_1 \times \overline{C}_2, \mathbb{Q}(1)) \rightarrow H^4(\overline{C}_1 \times \overline{C}_2, \mathbb{Q}(2))$ is now a subHodge structure of $H^2(\overline{C}_1 \times \overline{C}_2, \mathbb{Q}(1))$ with $V_{\mathbb{C}} = H^{2,0}(\overline{C}_1 \times \overline{C}_2) \oplus H^{0,2}(\overline{C}_1 \times \overline{C}_2)$.

Note that a simple $\overline{\mathbb{Q}}$ -spread argument implies that the \overline{C}_j in Example 4.8 are defined over $\overline{\mathbb{Q}}$ (and more precise statements are known classically). This leads to

Question 4.9. *Let W be a smooth projective surface. If $H^{2,0}(W) \oplus H^{0,2}(W)$ contains a non-trivial \mathbb{Q} -subHodge Structure of $H^2(W, \mathbb{Q})$, can W be obtained by base extension from a surface defined over $\overline{\mathbb{Q}}$?*

Proposition 4.10. *Suppose X is a K3 surface or an abelian surface. Then the answer to the previous question is positive.*

Proof. In both cases, $H^{0,2}$ is one dimensional, so the assumption implies that $H^{2,0}(W) \oplus H^{0,2}(W)$ arises from a \mathbb{Q} -subHodge structure of $H^2(W, \mathbb{Q})$. Since the former is a Hodge structure of type $(1, 0, 1)$, it follows that its Mumford-Tate group is abelian. In the case of an abelian surface, this implies that the Mumford-Tate group of the abelian variety is abelian, and therefore the abelian surface has complex multiplication. On the other hand, every CM abelian variety over \mathbb{C} is defined over number field. If W is a K3 surface, then it has maximal Picard rank, hence is well-known to be rigid, *a fortiori*, defined over $\overline{\mathbb{Q}}$. \square

Remark 4.11. *In spite of the mild “rigidity” assumption in Theorem 4.4, any attempt to extend the theorem to the generic point of S , without introducing some conjectural assumptions, seems incredibly difficult.*

5. GENERALITIES

Before proceeding further, and to be able to move further ahead, we explain some necessary assumptions. *For the remainder of this paper (with the occasional reminder), we assume the following:*

Assumptions 5.1. (i) The Hodge conjecture.

(ii) The Bloch-Beilinson conjecture on the injectivity of the Abel-Jacobi map for Chow groups of smooth projective varieties defined over $\overline{\mathbb{Q}}$ (see [Lew2, Conj 3.1 and §4]).

To spare the reader of time consuming search of multiple sources by many others, we refer mostly to [Lew2], for all the necessary statements and details. Let Z/\mathbb{C} be any smooth projective variety of dimension d_0 , $r \geq 0$, and put

$$\text{CH}_{AJ}^r(Z; \mathbb{Q}) = \ker (AJ : \text{CH}_{\text{hom}}^r(Z; \mathbb{Q}) \rightarrow J(H^{2r-1}(Z, \mathbb{Q}(r))).$$

As mentioned in §1, we recall the notion of a descending Bloch-Beilinson (BB) filtration $\{F^\nu \text{CH}^r(Z; \mathbb{Q})\}_{\nu \geq 0}$, with $F^0 \text{CH}^r(Z; \mathbb{Q}) = \text{CH}^r(Z; \mathbb{Q})$, $F^1 \text{CH}^r(Z; \mathbb{Q}) = \text{CH}_{\text{hom}}^r(Z; \mathbb{Q})$, $F^{r+1} \text{CH}^r(Z; \mathbb{Q}) = 0$, and satisfying a number of properties codified for example in [Ja2, §11], [Lew2, §4]. There is also the explicit construction of a candidate BB filtration by Murre, based on a conjectured Chow-Künneth decomposition, and subsequent conjectures in [M], which is equivalent to the existence of

a BB filtration as formulated in [Ja2, §11]). Further, Jannsen also proved that the BB filtration is unique if it exists. The construction of the filtration in [Lew1] (and used in [Lew2]) relies on Assumptions 5.1, which if true, provides the existence of a BB filtration, and hence is the same filtration as Murre's, by the aforementioned uniqueness.

All candidate filtrations seem to show that $F^2\mathrm{CH}^r(Z; \mathbb{Q}) \subseteq \mathrm{CH}_{AJ}^r(Z; \mathbb{Q})$. The following is considered highly non-trivial:

Conjecture 5.2. $\mathrm{CH}_{AJ}^r(Z; \mathbb{Q}) = F^2\mathrm{CH}^r(Z; \mathbb{Q})$.

In light of Assumptions 5.1, this is equivalent to the surjectivity of

$$\mathrm{cl}_{1,r} : \mathrm{CH}^r(\mathrm{Spec}(\mathbb{C}(Z)), 1; \mathbb{Q}) \rightarrow \Gamma(H^{2r-1}(\mathbb{C}(Z), \mathbb{Q}(r)))$$

as in [dJ-L, (S3)], provided both statements apply to all smooth projective Z/\mathbb{C} . For a proof, see [Lew2, Thm 1.1].

One of the key properties of the BB filtration is the factorization of graded pieces of that filtration through the Grothendieck motive. Let Δ_Z in $\mathrm{CH}^{d_0}(Z \times Z)$ be the diagonal class, with cohomology class $[\Delta_Z]$ in $H^{2d_0}(Z \times Z, \mathbb{Z}(d_0))$. Write $\Delta_Z = \sum_{p+q=2d_0} \Delta_Z(p, q)$ in $\mathrm{CH}^{d_0}(Z \times Z; \mathbb{Q})$ such that

$$\bigoplus_{p+q=2d_0} [\Delta_Z(p, q)] \in \bigoplus_{p+q=2d_0} H^p(Z, \mathbb{Q}) \otimes H^q(Z, \mathbb{Q})(d_0) = H^{2d_0}(Z \times Z, \mathbb{Q}(d_0)).$$

is the Künneth decomposition of $[\Delta_Z]$. Then

$$\Delta_Z(p, q)_* \big|_{Gr_F^\nu \mathrm{CH}^r(Z; \mathbb{Q})}$$

is independent of the choice of $\Delta_Z(p, q)$. (This is essentially due to the fact that $F^1\mathrm{CH}^r(Z; \mathbb{Q}) = \mathrm{CH}_{\mathrm{hom}}^r(Z; \mathbb{Q})$, and functoriality properties of the BB filtration.) Furthermore,

$$(5.3) \quad \Delta_Z(2d_0 - 2r + \ell, 2r - \ell)_* \big|_{Gr_F^\nu \mathrm{CH}^r(Z; \mathbb{Q})} = \delta_{\ell, \nu} \mathrm{id}_{Gr_F^\nu \mathrm{CH}^r(Z; \mathbb{Q})}$$

with $\delta_{i,j}$ the Kronecker delta. Consequently,

$$(5.4) \quad \Delta_Z(2d_0 - 2r + \nu, 2r - \nu)_* \mathrm{CH}^r(X; \mathbb{Q}) \simeq Gr_F^\nu \mathrm{CH}^r(X; \mathbb{Q}),$$

and accordingly there is a non-canonical decomposition

$$\mathrm{CH}^r(Z; \mathbb{Q}) = \bigoplus_{\nu=0}^r \Delta_Z(2d_0 - 2r + \nu, 2r - \nu)_* \mathrm{CH}^r(Z; \mathbb{Q}).$$

In summary, we will view the kernel of the Abel-Jacobi map in terms of $Gr_F^\nu \mathrm{CH}^r(Z; \mathbb{Q})$ for $\nu \geq 2$, i.e., under Conjecture 5.2,

$$\mathrm{CH}_{AJ}^r(Z; \mathbb{Q}) = F^2\mathrm{CH}^r(Z; \mathbb{Q}) \simeq \bigoplus_{\nu=2}^r Gr_F^\nu \mathrm{CH}^r(Z; \mathbb{Q}),$$

(to re-iterate, non-canonically).

Remark 5.5. *Murre's idea [M] is that one can choose the $\Delta_Z(p, q)$ to be commuting, pairwise orthogonal idempotents. By Beilinson and Jannsen, such a lift is possible if $\mathrm{CH}_{\mathrm{hom}}^{d_0}(Z \times Z; \mathbb{Q})$ is a nilpotent ideal under composition, which is a consequence of Assumptions 5.1. It should be pointed out that such $\Delta_Z(p, q)$'s are still not unique.*

The following will play an important role in Section 6.

Proposition 5.6. *Under Assumptions 5.1, the map*

$$\Xi_{Z,*} := \bigoplus_{\nu=2}^r \Delta_Z(2d_0 - 2r + \nu, 2r - \nu)_* : F^2\mathrm{CH}^r(Z; \mathbb{Q}) \rightarrow F^2\mathrm{CH}^r(Z; \mathbb{Q}),$$

is an isomorphism. Moreover, if the $\Delta_Z(p, q)$'s are chosen as in Murre's Chow-Künneth decomposition (viz., in Remark 5.5), then it is the identity.

Proof. If $r = 2$, this is obvious, as $F^r\mathrm{CH}^r(Z; \mathbb{Q}) = \mathrm{Gr}_F^r\mathrm{CH}^r(Z; \mathbb{Q})$. So assume $r > 2$. The 5-lemma, together with the diagram

$$\begin{array}{ccccccc} & & F^r\mathrm{CH}^r(Z; \mathbb{Q}) & & & & \\ & & \parallel & & & & \\ 0 & \rightarrow & \mathrm{Gr}_F^r\mathrm{CH}^r(Z; \mathbb{Q}) & \rightarrow & F^{r-1}\mathrm{CH}^r(Z; \mathbb{Q}) & \rightarrow & \mathrm{Gr}_F^{r-1}\mathrm{CH}^r(Z; \mathbb{Q}) \rightarrow 0 \\ & & \Xi_{Z,*} \downarrow \parallel & & \Xi_{Z,*} \downarrow & & \Xi_{Z,*} \downarrow \parallel \\ 0 & \rightarrow & \mathrm{Gr}_F^r\mathrm{CH}^r(Z; \mathbb{Q}) & \rightarrow & F^{r-1}\mathrm{CH}^r(Z; \mathbb{Q}) & \rightarrow & \mathrm{Gr}_F^{r-1}\mathrm{CH}^r(Z; \mathbb{Q}) \rightarrow 0 \end{array}$$

tells us that the middle vertical arrow is an isomorphism. By an inductive-recursive argument, we arrive at another 5-lemma argument:

$$\begin{array}{ccccccc} 0 & \rightarrow & F^3\mathrm{CH}^r(Z; \mathbb{Q}) & \rightarrow & F^2\mathrm{CH}^r(Z; \mathbb{Q}) & \rightarrow & \mathrm{Gr}_F^2\mathrm{CH}^r(Z; \mathbb{Q}) \rightarrow 0 \\ & & \Xi_{Z,*} \downarrow \wr & & \Xi_{Z,*} \downarrow & & \Xi_{Z,*} \downarrow \parallel \\ 0 & \rightarrow & F^3\mathrm{CH}^r(Z; \mathbb{Q}) & \rightarrow & F^2\mathrm{CH}^r(Z; \mathbb{Q}) & \rightarrow & \mathrm{Gr}_F^2\mathrm{CH}^r(Z; \mathbb{Q}) \rightarrow 0 \end{array}$$

which implies the isomorphism in the proposition.

For the second statement, clearly $\Delta_Z = \bigoplus_{\ell=2r-2d_0}^{2r} \Delta_Z(2d_0 - 2r + \ell, 2r - \ell)$ induces the identity. But from (5.4) we see that the terms with $\ell \geq r + 1$ do not contribute because $\mathrm{Gr}_F^\ell\mathrm{CH}^r(Z; \mathbb{Q}) = 0$. Also, from (5.3) we find that if $\ell \neq \nu$, then $\Delta_Z(2d_0 - 2r + \ell, 2r - \ell)$ maps $F^\nu\mathrm{CH}^r(Z; \mathbb{Q})$ into $F^{\nu+1}\mathrm{CH}^r(Z; \mathbb{Q})$. But if $\ell < \nu$ and this correspondence is idempotent then we can iterate this, finding it kills $F^\nu\mathrm{CH}^r(Z; \mathbb{Q})$ as $F^{r+1}\mathrm{CH}^r(Z; \mathbb{Q}) = 0$. Taking $\nu = 2$, we are done. \square

So to understand more about $\mathrm{CH}_{AJ}^r(Z; \mathbb{Q})$, it makes sense to study

$$\Delta_Z(2d_0 - 2r + \nu, 2r - \nu)_*\mathrm{CH}_{AJ}^r(Z; \mathbb{Q}),$$

for $2 \leq \nu \leq r$.

6. MAIN RESULTS

In this section, we will be assuming Assumptions 5.1 as well as Conjecture 5.2. We shall also assume that we are working with Murre's Chow-Künneth decomposition. Furthermore, \bar{S} and X are assumed smooth and projective, with $\dim \bar{S} = N$, $\dim X = d$. Then we have the results of Section 5 for $Z = \bar{S} \times X$, with dimension $d_0 = N + d$.

Remark 6.1. *This remark is critical to understanding our approach to the main results in the remainder of this paper. We want to take some earlier results, in particular Proposition 5.6, one step further. Let us assume the notation and setting in Proposition 3.6, for $\overline{X} = \overline{S} \times X$, with \mathcal{K} as in (3.5). Our goal is to show that the RHS of Proposition 3.6 is zero; in particular, that*

$$\Xi_0 := \ker (AJ : \mathcal{K} \rightarrow J(H^{2r-1}(\overline{S} \times X, \mathbb{Q}(r))))$$

is contained in $N_{\overline{S}}^1 \text{CH}^r(\overline{S} \times X; \mathbb{Q}(r))$. Proposition 5.6 shows that the induced map

$$(6.2) \quad \bigoplus_{\nu=2}^r \Delta_{\overline{S} \times X}(2(N+d) - 2r + \nu, 2r - \nu)_* : \Xi_0 \rightarrow \Xi_0,$$

is the identity.

Below we only consider $\ell \leq N$ since ultimately we will be passing to the generic point $\eta_{\overline{S}}$ of \overline{S} , hence through an affine $S \subset \overline{S}$, where we apply the affine weak Lefschetz theorem, with $D = \overline{S} \setminus S$. Although not needed, we could also take $\ell \geq 2$ because the cases $\ell = 0, 1$ can be proved as in the proof of Proposition 4.1(2).

Note that

$$(6.3) \quad \Gamma(H^{2r-1}(S \times X, \mathbb{Q}(r))) = \bigoplus_{\ell=1}^N \Gamma\left(H^\ell(S, \mathbb{Q}(1)) \otimes H^{2r-\ell-1}(X, \mathbb{Q}(r-1))\right),$$

hence we break down our arguments involving each of the N terms on the RHS of (6.3). This is similar to how we handled things in §4. Note that if we apply the Künneth projector $[\Delta_{\overline{S}} \otimes \Delta_X(2d - 2r + \ell + 1, 2r - \ell - 1)]_*$ to the short exact sequence in (3.1) of §3 (with $\mathcal{Y} = D \times X$), where the action of the aforementioned Künneth projector on $H^{2r-1}(S \times X, \mathbb{Q}(r))$ is given by $\text{Pr}_{13,*}(\text{Pr}_{23}^*[\Delta_X(2d - 2r + \ell + 1, 2r - \ell - 1)] \cup \text{Pr}_{12}^*(-))$, observing that both $\text{Pr}_{13}, \text{Pr}_{12} : S \times X \times X \rightarrow S \times X$ are proper and flat, we end up with the short exact sequence:

$$\begin{aligned} 0 \rightarrow \left\{ \frac{H^\ell(\overline{S}, \mathbb{Q}(1))}{H_D^\ell(\overline{S}, \mathbb{Q}(1))} \right\} \otimes H^{2r-\ell-1}(X, \mathbb{Q}(r-1)) &\rightarrow H^\ell(S, \mathbb{Q}(1)) \otimes H^{2r-\ell-1}(X, \mathbb{Q}(r-1)) \\ &\rightarrow H_D^{\ell+1}(\overline{S}, \mathbb{Q}(1))^\circ \otimes H^{2r-\ell-1}(X, \mathbb{Q}(r-1)) \rightarrow 0, \end{aligned}$$

where

$$H_D^{\ell+1}(\overline{S}, \mathbb{Q})^\circ = \ker (H_D^{\ell+1}(\overline{S}, \mathbb{Q}) \rightarrow H^{\ell+1}(\overline{S}, \mathbb{Q})).$$

This accordingly modifies the bottom row of (3.2) of §3 in the obvious way. As a reminder, the Künneth components cycle representatives $\Delta_X(2d - \bullet, \bullet)$ of the diagonal class Δ_X (and of $\Delta_{\overline{S}}$, hence the product $\Delta_{\overline{S} \times X} = \Delta_{\overline{S}} \otimes \Delta_X$) are now assumed chosen in the sense of Murre (see Remark 5.5). So we can likewise apply the projector $\Delta_{\overline{S}} \otimes \Delta_X(2d - 2r + \ell + 1, 2r - \ell - 1)$ to the top row of (3.2) of §3, and arrive at a modified commutative diagram (3.2), based on $\ell = 1, \dots, N$. We can be more explicit here. For the sake of brevity, let us denote $\Delta_X(2d - 2r + \ell + 1, 2r - \ell - 1)$ in $\text{CH}^d(X \times X; \mathbb{Q})$ here with P , and let $\mathcal{Y} = D \times X$ for some codimension one subscheme $D \subset \overline{S}$. Then $\Delta_{\overline{S}} \otimes P$ acts on $\text{CH}^r(\overline{S} \times X, 1; \mathbb{Q})$ in a natural way, and its action on $\text{CH}_{D \times X}^r(\overline{S} \times X; \mathbb{Q})$ is given as follows. First of all, $\text{CH}_{D \times X}^r(\overline{S} \times X; \mathbb{Q}) = \text{CH}^{r-1}(D \times X; \mathbb{Q})$. There are proper flat maps $D \times X \times X$ given by projections $\text{Pr}_{12} : D \times X \times X \rightarrow D \times X$, $\text{Pr}_{23} : D \times X \times X \rightarrow X \times X$ and $\text{Pr}_{13} : D \times X \times X \rightarrow D \times X$. For γ in $\text{CH}^{r-1}(D \times X; \mathbb{Q})$, $\text{Pr}_{12}^*(\gamma)$ in $\text{CH}^{r-1}(D \times X \times X; \mathbb{Q})$ is defined (flat pullback). For $P \in \text{CH}^d(X \times X; \mathbb{Q})$, the intersection $\text{Pr}_{12}^*(\gamma) \bullet \text{Pr}_{23}^*(P) \in$

$\mathrm{CH}^{r-1+d}(D \times X \times X; \mathbb{Q})$ is likewise well defined [F, §2]. The action then is given by $Pr_{13,*}(Pr_{12}^*(\gamma) \bullet Pr_{23}^*(P))$ in $\mathrm{CH}^{r-1}(D \times X; \mathbb{Q}) = \mathrm{CH}_{D \times X}^r(\overline{S} \times X; \mathbb{Q})$. The action of $\Delta_{\overline{S}} \otimes P$ on $\mathrm{CH}_{\mathrm{hom}}^r(\overline{S} \times X; \mathbb{Q})$ is clear. Finally, by an elementary Hodge theory argument, one arrives at a modified version of Proposition 3.6. Specifically, $\Delta_{\overline{S}} \otimes P$ acts naturally on all terms on the RHS of the display in Proposition 3.6, making use of functoriality of the Abel-Jacobi map, and operates naturally on the LHS, as clearly evident in the above discussion following (6.3). As ℓ ranges from $1, \dots, N$, both sides of the aforementioned display decompose accordingly into a direct sum.

We need to determine what $\Delta_{\overline{S}} \otimes \Delta_X(2d - 2r + \ell + 1, 2r - \ell - 1)$ does to γ , which is an algebraic cycle of codimension r (dimension $N + d - r$) on $\overline{S} \times X$, but supported on $D \times X$. Decomposing $\Delta_{\overline{S}}$, we can write

$$\Delta_{\overline{S} \times X}(2(N + d) - 2r + \nu, 2r - \nu),$$

as a sum of

$$(6.4) \quad \Delta_{\overline{S}}(2N + \nu - \ell - 1, \ell + 1 - \nu) \otimes \Delta_X(2d - 2r + \ell + 1, 2r - \ell - 1).$$

We recall that we have $2 \leq \nu \leq r$, and, as indicated earlier, only consider ℓ with $2 \leq \ell \leq N$. Before stating our next result, it is helpful to introduce the following, which includes those \overline{S} which are complete intersections in projective space or more generally a Grassmannian.

Lemma 6.5. *Suppose that \overline{S} is a variety of dimension N such that for $i \neq N$:*

$H^i(\overline{S}, \mathbb{Q})$ is zero for i odd and generated by algebraic cycles for i even.

Then \overline{S} admits a Chow-Künneth decomposition in the sense of [M], Remark 5.5 with the supports of the Künneth projectors compatible with the supports of the cohomology classes in $H^\bullet(\overline{S}, \mathbb{Q})$. Specifically, for $j \neq N$, $\Delta_{\overline{S}}(2N - 2j, 2j)$ is contained in the image of $\mathrm{CH}^{N-j}(\overline{S}; \mathbb{Q}) \otimes \mathrm{CH}^j(\overline{S}; \mathbb{Q})$ under pullback to $\mathrm{CH}^r(\overline{S} \times \overline{S}; \mathbb{Q})$ and taking the product..

Proof. For $j = 0, \dots, N$, let W_{2N-2j} in $\mathrm{CH}^{N-j}(\overline{S}, \mathbb{Q})$ and V_{2j} in $\mathrm{CH}^j(\overline{S}, \mathbb{Q})$ be algebraic cycles such that $\lambda_j := \deg(\langle W_{2N-2j}, V_{2j} \rangle_{\overline{S}}) \neq 0$. Then

$$\{W_{2N-2k} \times V_{2k}\} \circ \{W_{2N-2j} \times V_{2j}\} = \begin{cases} 0 & \text{if } k \neq j \\ \lambda_j \cdot \{W_{2N-2j} \times V_{2j}\} & \text{if } k = j \end{cases}$$

To see this, compute $\{W_{2N-2k} \times V_{2k}\} \circ \{W_{2N-2j} \times V_{2j}\}$ as

$$(6.6) \quad Pr_{13,*}(\langle Pr_{12}^*(W_{2N-2k} \times V_{2k}), Pr_{23}^*(W_{2N-2j} \times V_{2j}) \rangle_{\overline{S} \times \overline{S} \times \overline{S}})$$

in $\mathrm{CH}^N(\overline{S} \times \overline{S}; \mathbb{Q})$. If $j = k$ the statement is clear. If $k > j$ then $V_{2k} \cap W_{2N-2j} = 0$ by codimension, and if $k < j$ then this intersection has dimension at least 1, so (6.6) has codimension bigger than N . In either case it is trivial. This principle allows us to define, for $i \neq N$, mutually orthogonal idempotents π_i in $\mathrm{CH}^N(\overline{S} \times \overline{S}; \mathbb{Q})$ with $[\pi_i]$ in $H^{2N-i}(\overline{S}, \mathbb{Q}) \otimes H^i(\overline{S}, \mathbb{Q})(N)$. We can take $\pi_i = 0$ if $i \neq N$ odd, so that $[\pi_i] = [\Delta(2N - i, i)] = 0$ by our assumption on the odd cohomology groups. For even $i = 2j \neq N$, we can arrange that $[\pi_{2j}] = [\Delta(2N - 2j, 2j)]$ by using the assumption about the even cohomology groups being generated by algebraic cocycles, as well as the non-degeneracy of the intersection product. Put $\pi_N = \Delta_{\overline{S}} - \sum_{i \neq N} \pi_i$. Because the π_i for $i \neq N$ are mutually orthogonal idempotents by construction, the same

holds if we use all π_0, \dots, π_{2N} . Because $[\pi_i] = [\Delta_{\overline{S}}(2N - i, i)]$ for $i \neq N$, it follows from the definition of π_N that $[\pi_N] = [\Delta_{\overline{S}}(N, N)]$ as well. \square

Theorem 6.7. *Suppose that \overline{S} is a variety of dimension N such that for $i \neq N$:*

$H^i(\overline{S}, \mathbb{Q})$ is zero for i odd and generated by algebraic cycles for i even.

Given Assumptions 5.1 and Conjecture 5.2, then

$$\mathrm{CH}^r(X_{\eta_{\overline{S}}}, 1; \mathbb{Q}) \rightarrow \Gamma(H^{2r-1}(X_{\eta_{\overline{S}}}, \mathbb{Q}(r))),$$

is surjective.

Proof. We shall use the Chow-Künneth decomposition of $\Delta_{\overline{S}}$ as in Lemma 6.5. Observe that with regard to (6.4),

$$\Delta_{\overline{S}}(2N + \nu - \ell - 1, \ell + 1 - \nu) = \Delta_{\overline{S}}(N, N) \Leftrightarrow \ell + 1 - \nu = N.$$

But $\ell \leq N$ and $\nu \geq 2$, so this never happens. Also, the situation where $\ell + 1 - \nu = 0$ does not contribute. Namely, remember that $\gamma \in \mathrm{CH}_{D \times X}^r(\overline{S} \times X; \mathbb{Q})^\circ$ maps to a class in $\mathrm{CH}_{AJ}^r(\overline{S} \times X; \mathbb{Q})$. Since $|\gamma| \subset D \times X$, $\Delta_{\overline{S}}(2N, 0) = \{p\} \times \overline{S}$ for some $p \in \overline{S}$, so that $D \times \overline{S}$ doesn't meet $\Delta_{\overline{S}}(2N, 0)$ for a suitable choice of p , it follows in this case that

$$(\Delta_{\overline{S}}(2N, 0) \otimes \Delta_X(2d - 2r + \nu, 2r - \nu))_*(\gamma) = 0.$$

For $1 \leq \ell + 1 - \nu \leq N - 1$, and γ in \mathcal{K} , we have that

$$(\Delta_{\overline{S}}(2N + \nu - \ell - 1, \ell + 1 - \nu) \otimes \Delta_X(2d - 2r + \ell + 1, 2r - \ell - 1))_*(\gamma)$$

is in $N_{\overline{S}}^1 \mathrm{CH}^r(\overline{S} \times X; \mathbb{Q})$. This is immediate from the fact that γ is null-homologous on $\overline{S} \times X$ and the support of the Chow-Künneth components here, as described in Lemma 6.5. Hence by Proposition 3.6 (more precisely, the incarnation of Proposition 3.6 in the discussion following (6.3)), and (6.2) of Remark 6.1,

$$\mathrm{CH}^r(X_{\eta_{\overline{S}}}, 1; \mathbb{Q}) \rightarrow \Gamma(H^{2r-1}(X_{\eta_{\overline{S}}}, \mathbb{Q}(r))),$$

is surjective. \square

6.1. Grand finale. For our final main result, we again consider $\overline{S} = \overline{C}_1 \times \dots \times \overline{C}_N$, a product of smooth complete curves (cf. Section 4.2). As before, we restrict ourselves to $2 \leq \ell \leq N$. Let us write

$$\Delta_{\overline{C}_j} = e_j \times \overline{C}_j + \Delta_{\overline{C}_j}(1, 1) + \overline{C}_j \times e_j,$$

where $e_j \in \overline{C}_j$ and $\Delta_{\overline{C}_j}(1, 1)$ is defined by the equality. Consider the decomposition

$$(6.8) \quad \Delta_{\overline{S}} = \Delta_{\overline{C}_1} \otimes \dots \otimes \Delta_{\overline{C}_N} = \bigotimes_{j=1}^N \{e_j \times \overline{C}_j + \Delta_{\overline{C}_j}(1, 1) + \overline{C}_j \times e_j\}.$$

Note that $\Delta_{\overline{S}}(D) = D$. It needs to be determined what the RHS of (6.8) does to D , and more precisely, what $\Delta_{\overline{S}} \otimes \Delta_X(2d - 2r + \ell + 1, 2r - \ell - 1)$ does to γ , which is an algebraic cycle of dimension $N + d - r$ supported on $D \times X$. Now up to permutation, the RHS of (6.8) is made up of terms of the form

$$(6.9) \quad (\Delta_{\overline{C}_j}(1, 1))^{\otimes_{j=1}^{k_1}} \otimes (\{e_j \times \overline{C}_j\})^{\otimes_{j=k_1+1}^{k_1+k_2}} \otimes (\{\overline{C}_j \times e_j\})^{\otimes_{j=k_1+k_2+1}^N},$$

which is in the $(k_1 + 2k_2, 2N - k_1 - 2k_2)$ -component of $\Delta_{\overline{S}}$. Because in (6.4) we want $\Delta_{\overline{S}}(2N + \nu - \ell - 1, \ell + 1 - \nu)$, we have

$$(6.10) \quad 2N + \nu - \ell - 1 = k_1 + 2k_2.$$

Clearly, we have $0 \leq k_1 + k_2 \leq N$, $2 \leq \nu \leq r$, and we are restricting ourselves to $2 \leq \ell \leq N$. Notice that if $k_1 + k_2 < N$, we arrive at the situation where a correspondence in (6.8), which when tensored with Δ_X , takes γ to an element of $N_{\overline{S}}^1 \text{CH}^r(\overline{S} \times X; \mathbb{Q})$. If $k_1 + k_2 = N$, then from (6.10), $N + \nu - \ell - 1 = k_2 \leq N$, and hence $\nu \leq \ell + 1$. As in the proof of Theorem 6.7, we can ignore the case $\nu = \ell + 1$.

Theorem 6.11. *Under Assumption 5.1, and Conjecture 5.2, if $\overline{S} = \overline{C}_1 \times \cdots \times \overline{C}_N$ is a product of smooth complete curves and X a smooth projective variety, then for any $r \geq 1$,*

$$\text{CH}^r(X_{\eta_{\overline{S}}}, 1; \mathbb{Q}) \rightarrow \Gamma(H^{2r-1}(X_{\eta_{\overline{S}}}, \mathbb{Q}(r))),$$

is surjective.

Proof. We will prove this by induction on $N \geq 1$, the case $N = 1$ being part of Proposition 4.1(2). It will be crucial that no part of $\Delta_{\overline{S}}(N, N)$ occurs because $\ell \leq N$ and $\nu \geq 2$ imply $\ell + 1 - \nu < N$. We shall argue on the summands of $\Delta_{\overline{S}}$ that, up to a permutation, are as in (6.9). The reductions preceding the theorem allow us to assume $k_1 + k_2 = N$, and that $1 \leq k_2 \leq N - 1$. We see (6.9) is

$$\Xi := \Delta_{\overline{C}_1}(1, 1) \otimes \cdots \otimes \Delta_{\overline{C}_{k_1}}(1, 1) \otimes \{e_{k_1+1} \times \overline{C}_{k_1+1}\} \otimes \cdots \otimes \{e_N \times \overline{C}_N\}$$

which is in the $(N + k_2, N - k_2)$ -component of $\Delta_{\overline{S}}$. Now let $D \subset \overline{S}$ have codimension 1. By choosing $\{e_{k_1+1}, \dots, e_N\}$ appropriately, we can assume that

$$D' := |\Xi[D]| \subseteq E \times \overline{C}_{k_1+1} \times \cdots \times \overline{C}_N$$

where $E \subset \overline{C}_1 \times \cdots \times \overline{C}_{k_1}$ has codimension 1. Let $\gamma \in \text{CH}_{AJ}^r(\overline{S} \times X; \mathbb{Q}) = F^2 \text{CH}^r(\overline{S} \times X; \mathbb{Q})$, supported on $D \times X$, represent a class

$$[\gamma] \in \Gamma(H_D^{\ell+1}(\overline{S}, \mathbb{Q}(1))^\circ \otimes H^{2r-\ell-1}(X, \mathbb{Q}(r-1))),$$

where

$$H_D^{\ell+1}(\overline{S}, \mathbb{Q}(1))^\circ = \ker(H_D^{\ell+1}(\overline{S}, \mathbb{Q}(1)) \rightarrow H^{\ell+1}(\overline{S}, \mathbb{Q}(1))).$$

Then taking note of (6.4), together with functoriality of the Abel-Jacobi map,

$$\gamma' := \Xi_*(\gamma) \in \text{CH}_{AJ}^r(\overline{S} \times X; \mathbb{Q}),$$

is supported on $D' \times X$. Indeed, we have

$$[\gamma'] \in \Gamma(H_{D'}^{\ell+1}(\overline{S}, \mathbb{Q}(1))^\circ \otimes H^{2r-\ell-1}(X, \mathbb{Q}(r-1))).$$

By the properties of the BB filtration, and in light of Remark 6.1, we can reduce to the case where $\gamma = \gamma'$ and $D = D'$. Notice that

$$H_{D'}^{\ell+1}(\overline{S}, \mathbb{Q})^\circ = \bigoplus_{j=1}^{2k_1-1} H_E^{j+1}(\overline{C}_1 \times \cdots \times \overline{C}_{k_1}, \mathbb{Q})^\circ \otimes H^{\ell-j}(\overline{C}_{k_1+1} \times \cdots \times \overline{C}_N, \mathbb{Q}).$$

Thus we can reduce to $[\gamma']$ in

$$\Gamma(H_E^{j+1}(\overline{C}_1 \times \cdots \times \overline{C}_{k_1}, \mathbb{Q}(1))^\circ \otimes H^{\ell-j}(\overline{C}_{k_1+1} \times \cdots \times \overline{C}_N, \mathbb{Q}) \otimes H^{2r-\ell-1}(X, \mathbb{Q}(r-1))).$$

Now let's put $\overline{S}_0 = \overline{C}_1 \times \cdots \times \overline{C}_{k_1}$ and $X_0 = \overline{C}_{k_1+1} \times \cdots \times \overline{C}_N \times X$. Then $\overline{S}_0 \times X_0 = \overline{S} \times X$, and $\gamma' \in \text{CH}_{AJ}^r(\overline{S}_0 \times X_0; \mathbb{Q})$ is supported on $E \times X_0$. Further, $\dim \overline{S}_0 < \dim \overline{S}$. Then by Proposition 3.6, and induction on N ,

$$\gamma' \in N_{\overline{S}_0}^1 \text{CH}^r(\overline{S}_0 \times X_0; \mathbb{Q}) \subset N_{\overline{S}}^1 \text{CH}^r(\overline{S} \times X; \mathbb{Q}).$$

As mentioned above, the same applies to γ , and we are done. \square

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